Riemannian Velocity and Acceleration Tensors/Vectors in Cartesian Coordinates Based upon the Great metric Tensor

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Abstract

Geometric quantities in all orthogonal curvilinear coordinates in theoretical Physics Mathematics are built upon the Euclidean geometry which is found on a well known metric tensor called the Euclidean metric tensor. But with the discovery of a great metric tensor in spherical polar coordinates $(r, \theta, \varphi, x^0)$ *in all gravitational fields in nature [1] has made Riemannian geometry to be opened up for exploration and exploitation and hence its application in theoretical Physics and Mathematics. In this paper, we use great metric tensor to obtain the Riemannian Velocity and Acceleration tensors/ vectors for application in theoretical Physics and other related fields.*

Keywords: *Riemannian geometry, Great metric tensor, Riemannian Velocity, Riemannian Acceleration and Cartesian coordinates.*

1:**0 Introduction**

The development of Riemannian Revolution in Physics and Mathematics has migrated theoretical Physics completely from the foundation of the good but old Euclidean Geometry into the foundation of the modern and more complete geometry called the Riemannian Geometry. This geometry has eluded the whole world because it was not founded on a well known metric tensor before now. But with the discovery of the one and only one mathematically most simple and physically most natural and satisfactory metric tensor called the great metric tensor for all gravitational fields in nature by Professor S.X.K Howusu, in the year 2009, in his book entitled: "Riemannian Revolution in Mathematics ans Physics II'' [1]. Follows this new development, we had able to formulate some geometrical quantities based upon the Riemannian geometry in our previous publications [2,3]. In this paper, we use the Great Metric Tensor to formulate the Riemannian Velocity and Acceleration Tensors/ Vectors in Cartesian Coordinates for application in theoretical physics and other related fields.

2:0 **Theory**

The Cartesian coordinates (x, y, z, x^0) are defined in terms of the spherical polar coordinates $(r, \theta, \varphi, x^0)$ by [4,5]

$$x = rsin\theta cos\varphi \quad (1)$$

$$y = rsin\theta sin\varphi \quad (2)$$

$$z = rcos\theta \quad (3)$$

Here:

$$r = [x^2 + y^2 + z^2]^{\frac{1}{2}} \quad (4)$$

and

$$\theta = \cos^{-1}\left[\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right] \quad (5)$$

and

$$\phi = \tan^{-1}\left[\frac{y}{x}\right] \quad (6)$$

The great metric tensor for all gravitational fields in nature in spherical polar coordinates (r, θ, ϕ, x^0) is given as [1].

$$g_{00} = -\left(1 + \frac{2}{c^2}f\right) \quad (7)$$

$$g_{11} = \left(1 + \frac{2}{c^2}f\right)^{-1} \quad (8)$$

$$g_{22} = r^2 \quad (9)$$

$$g_{33} = r^2 \sin^2\theta \quad (10)$$

$$g_{uv} = 0\,; Otherwise \quad (11)$$

From the well know transformation equation given by the covariant tensor [6,7] and consequently, upon transformation by using (2.1)-(2.8) we obtained the Riemannian metric tensor for all gravitational fields in the Cartesian coordinates explicitly as [2]:

$$g_{uv} = h_{uv} + f_{uv} \quad (12)$$

where

$$h_{11} = h_{22} = h_{33} = 1 \quad (13)$$

$$f_{11} = \frac{x^2}{[x^2 + y^2 + z^2]}\sum_{n=1}^{\infty}\binom{-1}{n}\left(\frac{2}{c^2}\right)^n f^n \quad (14)$$

$$f_{12} = f_{21} = \frac{xy}{[x^2 + y^2 + z^2]}\sum_{n=1}^{\infty}\binom{-1}{n}\left(\frac{2}{c^2}\right)^n f^n = g_{21} \quad (15)$$

$$f_{13} = f_{31} = \frac{xz}{[x^2 + y^2 + z^2]}\sum_{n=1}^{\infty}\binom{-1}{n}\left(\frac{2}{c^2}\right)^n f^n \quad (16)$$

$$f_{22} = \frac{y^2}{[x^2 + y^2 + z^2]}\sum_{n=1}^{\infty}\binom{-1}{n}\left(\frac{2}{c^2}\right)^n f^n \quad (17)$$

$$f_{23} = f_{32} = \frac{yz}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n \quad (18)$$

$$f_{33} = \frac{z^2}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n \quad (19)$$

$$h_{oo} = -1 \quad (20)$$

$$f_{oo} = -\frac{2}{c^2} f^n \quad (21)$$

and

$$h_{uv} = 0 = f_{uv} \text{ ; } Otherwise \quad (22)$$

and the determinant of this metric tensor, denoted as g is given by

$$g = -1 \quad (23)$$

if $g_{uv}$is a covariant metric tensor, then according to tensor analysis the contravariant metric tensor for this Riemannian metric tensor denote as $g^{uv}$is given as:

$$g^{00} = -\left(1 + \frac{2}{c^2} f\right)^{-1} \quad (24)$$

$$g^{11} = \left[1 + \frac{(x^2 + y^2)}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n\right] \left(1 + \frac{2}{c^2} f\right) \quad (25)$$

$$g^{12} = \left[\frac{xy}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n\right] \left(1 + \frac{2}{c^2} f\right) = g^{21} \quad (26)$$

$$g^{13} = \left[\frac{xz}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n\right] \left(1 + \frac{2}{c^2} f\right) = g^{31} \quad (27)$$

$$g^{22} = \left[1 + \frac{(x^2 + z^2)}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n\right] \left(1 + \frac{2}{c^2} f\right) \quad (28)$$

and

$$g^{uv} = 0, otherwise \quad (29)$$

These metric tensors define Riemannian line element, Riemannian volume element, Riemannian gradient, Riemannian divergence, Riemannian curl and Riemannian Laplacian in Cartesian coordinates, according to the Theory of Tensor and Vector Analysis [6]. These quantities are necessary and sufficient for the derivation of the fields in all Cartesian distribution of mass, charge and current. Now for the derivation of the equation of motion for test particles in all gravitational fields, we shall derive the expression for Riemannian velocity and acceleration in Cartesian coordinates.

2:1 Great Riemannian Velocity Tensor/Vector in Cartesian Coordinates

According to the theory of tensor analysis, the linear velocity tensor in four dimensional space-time, U^{α} is given in all gravitational fields in all orthogonal curvilinear coordinates x^u by [Spiegel, 1974]:

$$U^{\alpha} = \frac{d}{d\tau} x^{\alpha} = \dot{x}^{\alpha} \qquad (30)$$

where τ is proper time and a dot denotes one differentiation with respect to time in Einstein Cartesian Coordinates $(x, y, z, x^0), u^0, u^1, u^2, and\ u^3$ are given as:

$$u^0 = c\dot{t} \qquad (31)$$

$$u^1 = \dot{x} \qquad (32)$$

$$u^2 = \dot{y} \qquad (33)$$

and

$$u^3 = \dot{z} \qquad (34)$$

It may be noted that in Minkwoski Cartesian coordinates, x^0 is given as;

$$u^0 = ic\dot{t} \qquad (35)$$

This is the Great Riemannian Linear Velocity tensor according to the theory of Tensor Analysis, the coordinates (x, y, z) is given as [7].

$$\underline{u}_R = \left[u_x, u_y, u_z, u_{x^0}\right] \qquad (36)$$

where

$$u_{x^0} = -c\left(1 - \frac{2}{c^2} f\right)^{\frac{1}{2}} \dot{t} \qquad (37)$$

$$u_x = \left(1 + \frac{x^2}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n\right)^{\frac{1}{2}} \dot{x} \qquad (38)$$

$$u_y = \left(1 + \frac{y^2}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n\right)^{\frac{1}{2}} \dot{y} \qquad (39)$$

and

$$u_z = \left(1 + \frac{z^2}{[x^2 + y^2 + z^2]} \sum_{n=1}^{\infty} \binom{-1}{n} \left(\frac{2}{c^2}\right)^n f^n\right)^{\frac{1}{2}} \dot{z} \qquad (40)$$

This is the great Riemannian Linear Velocity Vector.

2:1 Great Riemannian Acceleration Tensor/Vector in Cartesian Coordinates

Follows the development of Great Riemannian linear velocity tensor/vector, the Riemannian Linear acceleration tensor in 4-dimensional space-time, a_R^{α}, in gravitational fields in nature and all orthogonal curvilinear coordinates x^{α} is obtained by theory of tensor analysis as [7].

$$a_R^{\alpha} = \ddot{x}^{\alpha} + \Gamma_{uv}^{\alpha} \dot{x}^u \dot{x}^v \qquad (41)$$

where Γ_{uv}^{α} is the Christoffel symbols of the second kind (or coefficient of affine connection) Pseudo tensor and a dot denotes one differentiation with respect to proper time τ. The non-zero results of Γ_{uv}^{α} based upon the great metric tensor in Cartesian coordinates are given as

$$\Gamma_{00}^{0} = \frac{1}{2} g^{00} g_{00,0} \tag{42}$$

$$\Gamma_{01}^{0} = \Gamma_{10}^{0} = \frac{1}{2} g^{00} g_{00,1} \tag{43}$$

$$\Gamma_{02}^{0} = \Gamma_{20}^{0} = \frac{1}{2} g^{00} g_{00,2} \tag{44}$$

$$\Gamma_{11}^{0} = -\frac{1}{2} g^{00} g_{11,0} \tag{45}$$

$$\Gamma_{12}^{0} = \Gamma_{21}^{0} = -\frac{1}{2} g^{00} g_{12,0} \tag{46}$$

$$\Gamma_{13}^{0} = \Gamma_{31}^{0} = -\frac{1}{2} g^{00} g_{13,0} \tag{47}$$

$$\Gamma_{22}^{0} = -\frac{1}{2} g^{00} g_{22,0} \tag{48}$$

$$\Gamma_{23}^{0} = \Gamma_{32}^{0} = -\frac{1}{2} g^{00} g_{23,0} \tag{49}$$

$$\Gamma_{33}^{0} = -\frac{1}{2} g^{00} g_{33,0} \tag{50}$$

$$\Gamma_{00}^{1} = \frac{1}{2} \left(g^{11} g_{00,1} - g^{12} g_{00,2} - g^{13} g_{00,3} \right) \tag{51}$$

$$\Gamma_{01}^{1} = \Gamma_{10}^{1} = \frac{1}{2} \left(g^{11} g_{11,0} + g^{12} g_{12,0} + g^{13} g_{13,0} \right) \tag{52}$$

$$\Gamma_{02}^{1} = \Gamma_{02}^{1} = \frac{1}{2} \left(g^{11} g_{12,0} + g^{12} g_{22,0} + g^{13} g_{32,0} \right) \tag{53}$$

$$\Gamma_{03}^{1} = \Gamma_{03}^{1} = \frac{1}{2} \left(g^{11} g_{13,0} + g^{12} g_{23,0} + g^{13} g_{33,0} \right) \tag{54}$$

$$\Gamma_{11}^{1} = \frac{1}{2} \left(g^{11} g_{11,1} + g^{12} g_{12,1} - g^{12} g_{11,2} + g^{13} g_{13,1} - g^{13} g_{11,3} \right) \tag{55}$$

$$\Gamma_{12}^{1} = \Gamma_{21}^{1} = \frac{1}{2} \left(g^{11} g_{11,2} + g^{12} g_{22,1} + g^{13} g_{13,2} + g^{13} g_{32,1} - g^{13} g_{12,3} \right) \tag{56}$$

$$\Gamma_{13}^{1} = \Gamma_{31}^{1} = \frac{1}{2} \left(g^{11} g_{11,3} + g^{12} g_{12,3} + g^{12} g_{23,1} + g^{12} g_{13,3} - g^{13} g_{33,1} \right) \tag{57}$$

$$\Gamma_{22}^{1} = g^{11} g_{21,2} + g^{13} g_{23,2} - \frac{1}{2} \left(g^{11} g_{22,1} + g^{13} g_{22,3} - g^{12} g_{22,2} \right) \tag{58}$$

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2}\left(g^{11}g_{21,3} + g^{11}g_{13,2} - g^{11}g_{23,1} + g^{12}g_{22,3} + g^{13}g_{33,2}\right) \quad (59)$$

$$\Gamma^1_{33} = g^{11}g_{31,3} + g^{12}g_{32,3} - \frac{1}{2}\left(g^{11}g_{33,1} + g^{12}g_{33,2} - \frac{1}{2}g^{13}g_{33,3}\right) \quad (60)$$

$$\Gamma^2_{01} = -\frac{1}{2}\left(g^{21}g_{00,0} + g^{22}g_{00,2} + g^{23}g_{00,3}\right) \quad (61)$$

$$\Gamma^2_{01} = \Gamma^2_{10} = \frac{1}{2}\left(g^{21}g_{11,0} + g^{22}g_{21,0} - \frac{1}{2}g^{23}g_{31,0}\right) \quad (62)$$

$$\Gamma^2_{02} = \Gamma^0_{20} = \frac{1}{2}\left(g^{21}g_{12,0} + g^{22}g_{22,0} + g^{23}g_{32,0}\right) \quad (63)$$

$$\Gamma^2_{03} = \Gamma^2_{30} = \frac{1}{2}\left(g^{21}g_{13,0} + g^{22}g_{23,0} + g^{23}g_{33,0}\right) \quad (64)$$

$$\Gamma^2_{11} = \frac{1}{2}\left(g^{21}g_{11,1} - g^{22}g_{11,2} - g^{23}g_{11,3}\right) + g^{22}g_{12,1} \quad (65)$$

$$\Gamma^2_{12} = \Gamma^3_{21} = \frac{1}{2}\left(g^{21}g_{11,2} + g^{22}g_{22,1} + g^{23}g_{13,1} - g^{23}g_{12,3}\right) \quad (66)$$

$$\Gamma^2_{13} = \Gamma^2_{32} = \frac{1}{2}\left(g^{21}g_{11,3} + g^{22}g_{12,3} + g^{22}g_{23,1} + g^{23}g_{22,3} - g^{22}g_{32,2}\right) \quad (67)$$

$$\Gamma^2_{23} = \left(g^{21}g_{12,2} + g^{23}g_{22,2}\right) - \frac{1}{2}\left(g^{21}g_{22,1} + g^{23}g_{22,3} - g^{22}g_{32,2}\right) \quad (68)$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{2}\left(g^{21}g_{31,2} + g^{21}g_{12,3} - g^{21}g_{32,1} + g^{22}g_{22,3} + g^{23}g_{33,2} + g^{23}g_{32,2} + g^{23}g_{32,3}\right) \quad (69)$$

$$\Gamma^2_{33} = g^{21}g_{31,3} + \frac{1}{2}\left(g^{22}g_{32,2} + g^{23}g_{32,3} - g^{21}g_{33,1} - g^{22}g_{33,2}\right) \quad (70)$$

$$\Gamma^3_{00} = -\frac{1}{2}\left(g^{31}g_{00,1} + g^{32}g_{00,2} + g^{33}g_{00,3}\right) \quad (71)$$

$$\Gamma^3_{01} = \Gamma^3_{10} = \frac{1}{2}\left(g^{31}g_{11,0} + g^{32}g_{21,0} + g^{33}g_{31,0} + g^{33}g_{31,0}\right) \quad (72)$$

$$\Gamma^3_{02} = \Gamma^3_{20} = \frac{1}{2}\left(g^{31}g_{12,0} + g^{32}g_{22,0} + g^{33}g_{32,0}\right) \quad (73)$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2}\left(g^{31}g_{13,0} + g^{32}g_{23,0} + g^{33}g_{33,0}\right) \quad (74)$$

$$\Gamma^3_{03} = \frac{1}{2}\left(g^{31}g_{11,1} - g^{32}g_{11,2} - g^{33}g_{11,3}\right) + g^{32}g_{12,1} + g^{33}g_{13,1} \quad (75)$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}\left(g^{31}g_{11,2} + g^{32}g_{22,1} + g^{33}g_{13,2} + g^{33}g_{32,1} - g^{33}g_{12,3}\right) \quad (76)$$

$$\Gamma_{22}^{3} = \left(g^{31}g_{21,2} + g^{33}g_{32,2}\right) + \frac{1}{2}\left(g^{32}g_{22,2} + g^{32}g_{22,2} - g^{31}g_{22,1} - g^{33}g_{22,3}\right) \quad (77)$$

$$\Gamma_{23}^{3} = \Gamma_{32}^{3} = \frac{1}{2}\left(g^{31}g_{21,3} + g^{32}g_{13,2} - g^{31}g_{23,1} + g^{32}g_{22,3} + g^{33}g_{33,2}\right) \quad (78)$$

$$\Gamma_{33}^{3} = \frac{1}{2}\left(g^{31}g_{31,3} + g^{31}g_{33,1} - g^{32}g_{33,1} - g^{32}g_{33,2} + g^{33}g_{3,3}\right) + g^{32}g_{32,2} \quad (79)$$

$$\Gamma_{\mu\nu}^{\alpha} = 0\,; \quad Otherwise \quad (80)$$

It follows from (42) – (80) that:

$$a_R^0 = \ddot{x}^0 + \Gamma_{00}^0\dot{x}^0\dot{x}^0 + 2\Gamma_{01}^0\dot{x}^0\dot{x}^1 + 2\Gamma_{02}^0\dot{x}^0\dot{x}^2 + 2\Gamma_{13}^0\dot{x}^0\dot{x}^3 + 2\Gamma_{11}^0\dot{x}^1\dot{x}^1 + 2\Gamma_{13}^1\dot{x}^1\dot{x}^3 + \Gamma_{12}^0\dot{x}^1\dot{x}^2 + 2\Gamma_{13}^0\dot{x}^1\dot{x}^3 + \Gamma_{22}^0\dot{x}^2\dot{x}^2 + 2\Gamma_{23}^0\dot{x}^2\dot{x}^3 + \Gamma_{33}^0\dot{x}^3\dot{x}^3 \quad (81)$$

and

$$a_R^1 = \ddot{x}^1 + \Gamma_{00}^1\dot{x}^0\dot{x}^0 + 2\Gamma_{01}^2\dot{x}^0\dot{x}^1 + 2\Gamma_{02}^1\dot{x}^0\dot{x}^2 + 2\Gamma_{03}^2\dot{x}^0\dot{x}^3 + \Gamma_{11}^1\dot{x}^1\dot{x}^1 + 2\Gamma_{12}^1\dot{x}^1\dot{x}^2 + 2\Gamma_{23}^1\dot{x}^2\dot{x}^3 + \Gamma_{33}^1\dot{x}^3\dot{x}^3 \quad (82)$$

and

$$a_R^2 = \ddot{x}^2 + \Gamma_{00}^2\dot{x}^0\dot{x}^0 + 2\Gamma_{01}^2\dot{x}^0\dot{x}^1 + 2\Gamma_{02}^2\dot{x}^0\dot{x}^2 + 2\Gamma_{03}^3\dot{x}^0\dot{x}^3 + \Gamma_{11}^1\dot{x}^1\dot{x}^1 + 2\Gamma_{12}^1\dot{x}^1\dot{x}^2 + 2\Gamma_{13}^2\dot{x}^1\dot{x}^3 + \Gamma_{22}^1\dot{x}^2\dot{x}^2 + 2\Gamma_{23}^2\dot{x}^2\dot{x}^3 + \Gamma_{33}^2\dot{x}^3\dot{x}^3 \quad (83)$$

and

$$a_R^3 = \ddot{x}^3 + \Gamma_{00}^3\dot{x}^0\dot{x}^0 + 2\Gamma_{01}^3\dot{x}^0\dot{x}^1 + 2\Gamma_{02}^3\dot{x}^0\dot{x}^2 + 2\Gamma_{03}^0\dot{x}^0\dot{x}^3 + \Gamma_{11}^3\dot{x}^1\dot{x}^1 + 2\Gamma_{12}^3\dot{x}^1\dot{x}^2 + 2\Gamma_{13}^3\dot{x}^1\dot{x}^3 + \Gamma_{22}^3\dot{x}^2\dot{x}^2 + 2\Gamma_{23}^3\dot{x}^2\dot{x}^3 + \Gamma_{33}^3\dot{x}^3\dot{x}^3 \quad (84)$$

where in Einstein coordinates of space-times in Cartesian coordinates:

$$x^1 = x\,; \quad x^2 = y\,; \quad x^3 = z\,; \quad x^0 = ct \quad (85)$$

Equation (81) – (84) is called the Great Riemann Linear Acceleration Tensor.

Hence, the Great Riemannian Acceleration Vector $\underline{a}_R$, is defined as:

$$\underline{a} = \left[(a_R)_x, (a_R)_y, (a_R)_z, (a_R)_{x^0}\right] \quad (86)$$

where

$$(a_R)_{x^0} = (g_{00})^{\frac{1}{2}}[a_R^0] \quad (87)$$

$$(a_R)_x = (g_{11})^{\frac{1}{2}}[a_R^1] \quad (88)$$

$$(a_R)_y = (g_{22})^{\frac{1}{2}}[a_R^2] \quad (89)$$

and

$$(a_R)_z = (g_{33})^{\frac{1}{2}}[a_R^3] \quad (90)$$

Equation (87) – (90) is called the Great Riemannian Laplacian Acceleration vector for all gravitational field in nature in Cartesian coordinates.

3:0 Results and Discussions

In this paper we derived the component of the Great Riemannian Linear Velocity Tensor/Vector and the Great Riemannian Linear Acceleration Tensor/Vector in Cartesian Coordinates as (80) – (84) and (87) – (90) respectively.

These results obtained in this paper are necessary and sufficient for expressing all Riemannian Mechanical quantities in all gravitational fields in nature (Riemannian Linear Momentum, Riemannian kinetic Energy, Riemannian Lagrangian and Riemannian Hamiltonian) in terms of Cartesian coordinates.

4:0 Conclusions

The Great Riemannian Linear Velocity Vector (80) – (84) and the Great Riemannian Linear Acceleration Vector (87) – (90) obtained in this paper pave a way for expressing all Riemannian dynamical laws of motion (Newton’s Law, Lagrange’s law, Hamilton’s law, Einstein’s Special Relativistic law of motion and Schrodinger’s law of quantum mechanics) entirely in terms of Cartesian coordinates.

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